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Generalized synchronization of multidimensional chaotic systems in terms of symbolic CTQ-analysis

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Abstract. A new approach is proposed to the analysis of generalized synchronization of multidimensional chaotic systems. The approach is based on the symbolic analysis of discrete sequences in the basis of a finite T-alphabet. In fact, the symbols of the T-alphabet encode the shape (the geometric structure) of a trajectory of a dynamical system. Investigation of symbolic sequences allows one to diagnose various regimes of chaos synchronization, including generalized synchronization. The characteristics introduced allow one to detect and study the restructuring and intermittency behavior of attractors in systems (the time structure of synchronization). The measure of T-synchronization proposed is generalized without restrictions to complex ensembles of strongly nonstationary and nonidentical large-dimensional oscillators with arbitrary configuration and network (lattice) topology. The main features of the method are illustrated by an example.

Keywords: Chaotic systems, Generalized synchronization, Attractor's structure, Intermittency of synchronism, Symbolic CTQ-analysis.

1. Introduction

Synchronization is one of the fundamental concepts of the theory of nonlinear dynamics and chaos theory. This phenomenon is widespread in nature, science, engineering, and society [1]. One of important manifestations of this phenomenon is the synchronization of chaotic oscillations, which was experimentally observed in various physical applications (see [1–5] and references therein) such as radio oscillators, mechanical systems, lasers, electrochemical oscillators, plasma and gas discharge, and quantum systems. The study of this phenomenon is also very important from the viewpoint of its application to information transmission [6], cryptographic coding [7] with the use of deterministic chaotic oscillations, and quantum computation [3, 8].

There are several types of synchronization of chaotic oscillations [2]: generalized synchronization [9], complete synchronization [10], antisynchronization [11], lag synchronization [12], frequency synchronization [13], phase synchronization [14], time scale synchronization [15], and T-synchronization [16]. For each type, an appropriate analytic apparatus and

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diagnostic methods have been developed. Nevertheless, intensive investigations are being continued that are aimed, on the one hand, at the examination of different types of synchronization from unique positions and, on the other hand, at the search for new types of synchronous behavior that do not fall under the above-mentioned types. In spite of the long history of the study of synchronization of chaotic oscillations, many important problems in this field remain unsolved.

These include generalized synchronization in the form

$$\mathbf{y}(t) = \mathbf{F}\big[\mathbf{x}(t),\, \boldsymbol{\tau}\big], \tag{1.1}$$

where $\mathbf{x}$ and $\mathbf{y}$ are multidimensional synchronized systems, $\mathbf{F}$ is a function of generalized link between the systems, and $\boldsymbol{\tau}$ is a delay vector between the phase variables of the systems $\mathbf{x}$ and $\mathbf{y}$.

In this paper, we develop an original method for the diagnostics and quantitative measurement of the characteristics of generalized synchronization of chaotic systems, which is aimed at the integrated study of the time structure of synchronism through the analysis of the so-called T-synchronization [16, 17].

The method is based on the formalism of symbolic CTQ-analysis proposed by the present author [18, 19] (the abbreviation CTQ stands for three alphabets with which the method operates: C, T, and Q). One should note that symbolic dynamics, for all of its seeming external simplicity, is a very strongly substantiated tool for the analysis of nonlinear dynamical systems [20–22].

This article is an expanded version of the report [23].

2. The Symbolic CTQ-analysis

Denote a discrete dynamical system as a mapping

$$\mathbf{s}_{k+1} = \mathbf{f}\left(\mathbf{s}_k,\, \mathbf{p}\right) \tag{2.2}$$

with the following properties: $\mathbf{s} \in \mathrm{S} \subseteq \mathbb{R}^N$, $k \in \mathrm{K} \subseteq \mathbb{Z}$, $\mathbf{p} \in \mathrm{P} \subseteq \mathbb{R}^M$, $n \in \overline{1,\, N}$, $m \in \overline{1,\, M}$. In formula (2.2), $\mathbf{s}$ is a state variable of the system and $\mathbf{p}$ is a vector of parameters. With mapping (2.2), we associate its trajectory in the space $\mathrm{S} \times \mathrm{K}$, which has the form of a semisequence $\{\mathbf{s}_k\}_{k=1}^{K}$, $k \in \overline{1,\, K}$.

2.1. T-alphabet

Define the initial mapping, which encodes (in terms of the final T-alphabet) the shape of the n-th component of the sequence $\{\mathbf{s}_k\}_{k=1}^{K}$ [18, 19]:

$$\left\{\mathbf{s}_{k-1}^{(n)},\, \mathbf{s}_{k}^{(n)},\, \mathbf{s}_{k+1}^{(n)}\right\} \Rightarrow T_k^{\alpha\varphi}|_n, \quad T_k^{\alpha\varphi} = \left[T_k^{\alpha\varphi}|_1,\, \ldots,\, T_k^{\alpha\varphi}|_n,\, \ldots,\, T_k^{\alpha\varphi}|_N\right]. \tag{2.3}$$

The graphic diagrams illustrating the geometry of the symbols $T^{\alpha\varphi}|_n$ for the k-th sample and the n-th phase variable are shown in Figure 1.

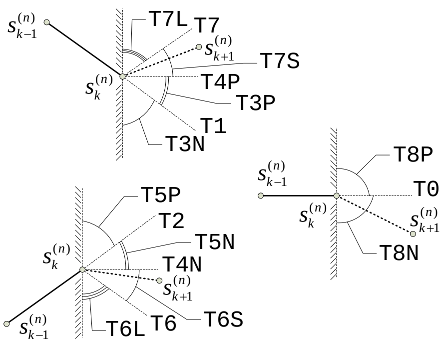

Figure 1. Geometry of T-alphabet symbols.

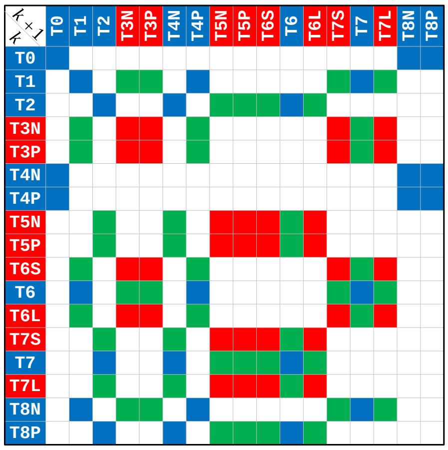

Figure 2. Transition matrix of T-alphabet symbols.

Strictly speaking, the mapping (2.3) is defined by the relations:

$$
\begin{array}{ll}
\texttt{T0} & \Delta s_- = \Delta s_+ = 0, \\
\texttt{T1} & \Delta s_- = \Delta s_+ < 0, \\
\texttt{T2} & \Delta s_- = \Delta s_+ > 0, \\
\texttt{T3N} & \Delta s_- < 0, \quad \Delta s_+ < \Delta s_-, \\
\texttt{T3P} & \Delta s_- < 0, \quad \Delta s_+ < 0, \quad \Delta s_+ > \Delta s_-, \\
\texttt{T4N} & \Delta s_- > 0, \quad \Delta s_+ = 0, \\
\texttt{T4P} & \Delta s_- < 0, \quad \Delta s_+ = 0, \\
\texttt{T5N} & \Delta s_- > 0, \quad \Delta s_+ > 0, \quad \Delta s_+ < \Delta s_-, \\
\texttt{T5P} & \Delta s_- > 0, \quad \Delta s_+ > \Delta s_-, \qquad\qquad , \\
\texttt{T6S} & \Delta s_- > 0, \quad \Delta s_+ < 0, \quad \Delta s_+ > -\Delta s_-, \\
\texttt{T6} & \Delta s_- = -\Delta s_+ > 0, \\
\texttt{T6L} & \Delta s_- > 0, \quad \Delta s_+ < 0, \quad \Delta s_+ < -\Delta s_-, \\
\texttt{T7S} & \Delta s_- < 0, \quad \Delta s_+ > 0, \quad \Delta s_+ < -\Delta s_-, \\
\texttt{T7} & \Delta s_- = -\Delta s_+ < 0, \\
\texttt{T7L} & \Delta s_- < 0, \quad \Delta s_+ > 0, \quad \Delta s_+ > -\Delta s_-, \\
\texttt{T8N} & \Delta s_- = 0, \quad \Delta s_+ < 0, \\
\texttt{T8P} & \Delta s_- = 0, \quad \Delta s_+ > 0.
\end{array}
\tag{2.4}
$$

Here $\Delta s_- = \mathbf{s}_k^{(n)} - \mathbf{s}_{k-1}^{(n)}$ and $\Delta s_+ = \mathbf{s}_{k+1}^{(n)} - \mathbf{s}_k^{(n)}$.

Thus, the T-alphabet includes the following set of symbols:

$$
\mathrm{T}_o^{\alpha\varphi} = \{\texttt{T0},\ \texttt{T1},\ \texttt{T2},\ \texttt{T3N},\ \texttt{T3P},\ \texttt{T4N},\ \texttt{T4P},\ \texttt{T5N},\ \texttt{T5P},\ \texttt{T6S},\ \texttt{T6},\ \texttt{T6L},\ \texttt{T7S},\ \texttt{T7},\ \texttt{T7L},\ \texttt{T8N},\ \texttt{T8P}\}. \tag{2.5}
$$

One can see from (2.5) that the symbol $T_k^{\alpha\varphi}|_n$ is encoded as $\texttt{T}\,i$, where i is the right-hand side of the symbol codes of the alphabet $\mathrm{T}_o^{\alpha\varphi}$. In turn, the symbol $T_k^{\alpha\varphi}$ is encoded in terms of $\texttt{T}\,i_1 \cdots i_n \cdots i_N$, see (2.3). The full alphabet $\mathrm{T}_o^{\alpha\varphi}|N$, which encodes the shape of the trajectory of the multidimensional sequence $\{\mathbf{s}_k\}_{k=1}^K$, consists of 17^N symbols.

2.2. Q-alphabet

In addition to the symbols $T_k^{\alpha\varphi}|_n$, we introduce the symbols $Q_k^{\alpha\varphi}|_n$:

$$Q_k^{\alpha\varphi}|_n \equiv T_k^{\alpha\varphi}|_n \to T_{k+1}^{\alpha\varphi}|_n, \quad Q_k^{\alpha\varphi} = [Q_k^{\alpha\varphi}|_1, \ldots, Q_k^{\alpha\varphi}|_n, \ldots, Q_k^{\alpha\varphi}|_N]. \tag{2.6}$$

All admissible transitions constitute a set of symbols of the alphabet $\mathrm{Q}_o^{\alpha\varphi} \ni Q_k^{\alpha\varphi}|_n$. These transitions are shown in Figure 2.

The symbol $Q_k^{\alpha\varphi}|_n$ is encoded as $\mathtt{Q}\, i\, j$, where i and j are the right-hand sides of the symbol codes of the alphabet $\mathrm{T}_o^{\alpha\varphi}$ for the states k and $k+1$, respectively. In turn, the symbol $Q_k^{\alpha\varphi}$ is encoded in terms of $\mathtt{Q}\, i_1 \cdots i_n \cdots i_N\, j_1 \cdots j_n \cdots j_N$, see (2.6). The full alphabet $\mathrm{Q}_o^{\alpha\varphi}|N$, which encodes the shape of the trajectory of the sequence $\{\mathbf{s}_k\}_{k=1}^K$, consists of 107^N symbols (see Figure 2).

2.3. Symbolic TQ-image of a dynamical system

Let us introduce a directed graph

$$\Gamma^{TQ} = \left\langle \mathrm{V}^\Gamma, \mathrm{E}^\Gamma \right\rangle, \quad \mathrm{V}^\Gamma \subseteq \mathrm{T}_o^{\alpha\varphi}|N, \quad \mathrm{E}^\Gamma \subseteq \mathrm{Q}_o^{\alpha\varphi}|N, \tag{2.7}$$

which is a complete symbolic TQ-image of the dynamical system (2.2). By definition, V^Γ is the vertex set and E^Γ is the edge set of Γ^{TQ}. According to its topology, the graph Γ^{TQ} has no multiple arcs but has loops.

By analogy with (2.7), introduce a directed graph

$$\Gamma^{TQ}|_n = \left\langle \mathrm{V}^\Gamma|_n, \mathrm{E}^\Gamma|_n \right\rangle, \quad \mathrm{V}^\Gamma|_n \subseteq \mathrm{T}_o^{\alpha\varphi}, \quad \mathrm{E}^\Gamma|_n \subseteq \mathrm{Q}_o^{\alpha\varphi}, \tag{2.8}$$

which is a particular symbolic TQ-image of the dynamical system with respect to its n-th phase variable. Denote the graph $\Gamma^{TQ}|_n$ corresponding to the full alphabets $\mathrm{T}_o^{\alpha\varphi}$ and $\mathrm{Q}_o^{\alpha\varphi}$ by Γ_o^{TQ}.

A particular symbolic TQ-image (2.8) can be obtained from the graph Γ^{TQ} by the gluing (or identification) of its vertices, redirection of the edges incident to these vertices, and the removal of the arising multiple edges.

Introduce the operation of gluing:

$$\Gamma^{TQ}|_n = \mathcal{R}^{\mathrm{TQ}}\Big(\Gamma^{TQ}, n\Big). \tag{2.9}$$

Let $v \in \mathrm{V}^\Gamma$, $e \in \mathrm{V}^\Gamma$, $v' \in \mathrm{V}^\Gamma|_n$, and $e' \in \mathrm{V}^\Gamma|_n$. Then

$$\begin{aligned} &v' \equiv \mathtt{T}\, i, && v \equiv \mathtt{T}\, i_1 \cdots i_n \cdots i_N, \\ &e' \equiv \mathtt{Q}\, i\, j, && e \equiv \mathtt{Q}\, i_1 \cdots i_n \cdots i_N\, j_1 \cdots j_n \cdots j_N. \end{aligned} \tag{2.10}$$

The gluing of vertices and edges is formally expressed by the following conditions:

$$\begin{aligned} &\mathrm{V}^\Gamma|_n \ni \mathtt{T}\, j : \mathtt{T}\, i_1 \cdots i_n \cdots i_N \in \mathrm{V}^\Gamma,\ i_n = j, \\ &\mathrm{E}^\Gamma|_n \ni \mathtt{Q}\, i\, j : \mathtt{Q}\, k_1 \cdots k_n \cdots k_N\, l_1 \cdots l_n \cdots l_N \in \mathrm{E}^\Gamma,\ k_n = i,\ l_n = j. \end{aligned} \tag{2.11}$$

The graph (2.8) can be weighted (on its vertices and edges) by the occurrence frequency of characters $*$ in the sequence $\{\mathbf{s}_k^{(n)}\}_{k=1}^K$:

$$\Delta^*|_n = \frac{|\mathrm{M}^*|_n|}{\left|\bigcup\limits_* \mathrm{M}^*|_n\right|}, \quad 0 \leqslant \Delta^*|_n \leqslant 1, \tag{2.12}$$

where $|\cdot|$ is the cardinality of the set and $*$ is a symbol of which the multiset $\mathrm{M}^*|_n$ consists:

$$\Delta^{\mathrm{T}}|_n : \mathrm{M}^*|_n \ni T_k^{\alpha\varphi}|_n : \, T_k^{\alpha\varphi}|_n \backslash \mathtt{T} = *, \; * \in \mathrm{T}_o^{\alpha\varphi} \backslash \mathtt{T}, \tag{2.13a}$$

$$\Delta^{\mathrm{Q}}|_n : \mathrm{M}^*|_n \ni Q_k^{\alpha\varphi}|_n : \, Q_k^{\alpha\varphi}|_n \backslash \mathtt{Q} = *, \; * \in \mathrm{Q}_o^{\alpha\varphi} \backslash \mathtt{Q}. \tag{2.13b}$$

Similar characteristics Δ^{T} and Δ^{Q} are determined for the graph Γ^{TQ}. Note that the calculation of Δ^{T} and Δ^{Q} allows one to quantitatively assess various properties of the trajectory of the sequence $\{\mathbf{s}_k\}_{k=1}^K$ in the space $\mathrm{S} \times \mathrm{K}$, including the Markov characteristic of the sequence $\{T_k^{\alpha\varphi}\}_{k=1}^K$ [20, 22].

3. T-synchronization

Let us make a remark. For simplicity, but without loss of generality, suppose that the time sequence $\{\mathbf{s}_k\}_{k=1}^K$ of dimension N is formed by a combination of the phase variables of N one-dimensional dynamical systems; i.e., suppose that $\mathbf{s}_k^{(n)}$ is the value of the phase variable of the nth system at the kth instant of time.

3.1. Complete T-synchronization

Definition. Dynamical systems are *synchronous* at time instant k in the sense of Complete T-synchronization [17] if the condition $J_k = 1$ is satisfied, where

$$J_k = \begin{cases} 1 & T_k^{\alpha\varphi}|_1 = \ldots = T_k^{\alpha\varphi}|_n = \ldots = T_k^{\alpha\varphi}|_N, \\ 0 & \text{otherwise.} \end{cases} . \tag{3.14}$$

Thus, $\{J_k\}_{k=1}^K$ is the indicator sequence of T-synchronization.

Taking into account possible antisynchronization [11] between the systems, we should also consider all possible variants of inversion of their phase variables: $\mathbf{s}_k^{(n)} \to -1 \cdot \mathbf{s}_k^{(n)}$. In this case, for the nth system, a change of symbols $T_k^{\alpha\varphi}|_n$ in the kth sample occurs according to the scheme

$$\begin{aligned} &\mathtt{T0} \leftrightarrow \mathtt{T0}, \\ &\mathtt{T1} \leftrightarrow \mathtt{T2}, \quad \mathtt{T3N} \leftrightarrow \mathtt{T5P}, \quad \mathtt{T3P} \leftrightarrow \mathtt{T5N}, \quad \mathtt{T4N} \leftrightarrow \mathtt{T4P}, \\ &\mathtt{T6S} \leftrightarrow \mathtt{T7S}, \quad \mathtt{T6} \leftrightarrow \mathtt{T7}, \quad \mathtt{T6L} \leftrightarrow \mathtt{T7L}, \quad \mathtt{T8N} \leftrightarrow \mathtt{T8P}. \end{aligned} \tag{3.15}$$

Denote the variants of inversion by number m. The total number of variants of inversion is $M = 2^{N-1}$.

Synchronization between systems can also be set in the lag regime [12]. To detect this synchronization, one should move a little the phase trajectories of the systems with respect to each other (the shifts are $h_n \geqslant 0$):

$$\left\{T_k^{\alpha\varphi}|_1 \to T_{k+h_1}^{\alpha\varphi}|_1, \, \ldots, \, T_k^{\alpha\varphi}|_n \to T_{k+h_n}^{\alpha\varphi}|_n, \, \ldots, \, T_k^{\alpha\varphi}|_N \to T_{k+h_N}^{\alpha\varphi}|_N\right\}. \tag{3.16}$$

The antisynchronization and lag synchronization regimes may coexist; therefore, when calculating a partial integral coefficient of synchronism, we take this fact into consideration:

$$\delta_{m,\mathbf{h}}^s = \frac{1}{K^* + 1 - k^*} \sum_{k=k^*}^{K^*} J_k |\, \{m, \, \mathbf{h}\}, \tag{3.17}$$

where $k^* = 1 + \max(h_1, \ldots, h_N)$, $K^* = K + \min(h_1, \ldots, h_N)$, and K is the length of the sequence $\{T_k^{\alpha\varphi}\}_{k=1}^K$.

On the basis of the partial coefficient, we calculate the total integral coefficient of synchronism of the systems:

$$\delta^s = \max_m \max_{\mathbf{h}} \delta^s_{m,\mathbf{h}}, \quad 0 \leqslant \delta^s \leqslant 1, \tag{3.18}$$

i.e., we take a combination of shifts between the trajectories of the systems and a variant of inversion of their phase variables that, taken together, provide the maximum number of samples k satisfying the condition $J_k = 1$.

4. Time structure of synchronization of chaotic systems

The quantity δ^s introduced in (3.18) characterizes the synchronism of the systems on average over a period of $t_K - t_1$. As mentioned in the Introduction, most investigations on the synchronization of chaos are usually restricted to this situation. However, often a researcher may be interested in the time structure of synchronization of systems. Recall that by this structure one means the spikes in the synchronous behavior of the phase variables of the systems between which the synchronism level is characterized by a small quantity, i.e., one means intermittent behavior [24, 25].

In [17], the present author introduced the concept of a synchronous domain SD – a set of samples of a time series that satisfy the condition ($\vee$ is the symbol of the logical operation OR)

$$\begin{gathered}\mathrm{SD}_r : \{J_{k'} = 1,\ J_{k''} = 0 \vee k'' = 0,\ J_{k'''} = 0 \vee k''' = K+1\}, \\ k' \in \overline{b_r^{\mathrm{SD}},\, b_r^{\mathrm{SD}} + L_r - 1}, \quad k'' = b_r^{\mathrm{SD}} - 1, \quad k''' = b_r^{\mathrm{SD}} + L_r^{\mathrm{SD}},\end{gathered} \tag{4.19}$$

where b_r^{SD}, L_r^{SD}, and r are the emergence time, the length, and the ordinal number of a synchronous domain, respectively. In this case, the following conditions are satisfied: $L_r^{\mathrm{SD}} \leqslant K$, and the total number of synchronous domains (in the original sequence) $R^{\mathrm{SD}} \leqslant (K+1)\operatorname{div} 2$.

To quantitatively describe the structure of synchronization of systems, the author introduced in [17] the spectral density function of synchronous domains SD:

$$H^{\mathrm{SD}}[L] = \sum_{r=1}^{R^{\mathrm{SD}}} \delta[L_r^{\mathrm{SD}}, L], \quad L \in \overline{1, K}, \tag{4.20}$$

where $\delta[\cdot, \cdot]$ is the Kronecker delta.

To analyze the degree of degeneracy of the structure of synchronous domains, we additionally define a quantity E^{SD} – the entropy of the structure of synchronous domains (according to Shannon) [16], which makes sense for $\delta^s > 0$:

$$E^{\mathrm{SD}} = -\sum_{i=1}^{K} P^{\mathrm{SD}}[i] \ln P^{\mathrm{SD}}[i], \quad P^{\mathrm{SD}}[L] = \frac{H^{\mathrm{SD}}[L]}{\sum\limits_{i=1}^{K} H^{\mathrm{SD}}[i]}. \tag{4.21}$$

It follows from Shannons entropy properties that the entropy E^{SD} is minimal ($E^{\mathrm{SD}} = 0$) when the spectrum $H^{\mathrm{SD}}[L]$ is degenerate (all synchronous domains have the same length) and

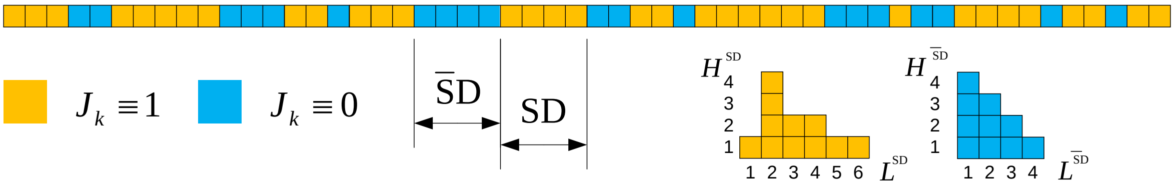

Figure 3. Basic characteristics of the time structure of synchronism.

maximal ($E^{\mathrm{SD}} = \hat{E}^{\mathrm{SD}}$) in the case of a uniform comb spectrum $H^{\mathrm{SD}}[L]$ with the maximum number of different lengths of synchronization domains equal to $\hat{W}^{\mathrm{SD}}_{cmb}$:

$$\hat{W}^{\mathrm{SD}}_{cmb} = \min\left\{ \left\lfloor \frac{\sqrt{1+8\,\delta^s K}-1}{2} \right\rfloor,\ K - \delta^s K + 1 \right\}, \quad \hat{E}^{\mathrm{SD}} = \ln \hat{W}^{\mathrm{SD}}_{cmb}, \tag{4.22}$$

where $\lfloor a \rfloor$ is the integer part of a.

On the basis of (4.21) and (4.22), we define the relative entropy of the structure of synchronous domains:

$$\Delta^{\mathrm{SD}}_E = \frac{E^{\mathrm{SD}}}{\hat{E}^{\mathrm{SD}}}. \tag{4.23}$$

It makes sense to apply the quantity Δ^{SD}_E when the researcher should compare synchronization cases with different values of δ^s and/or K.

Nevertheless, for the full description of the intermittent behavior of chaotic systems during synchronization, it is obviously insufficient to study only synchronous domains SD. To obtain a complete and closed idea of the time structure of synchronism of dynamical systems (a complete and closed representation of the intermittency structure), in [16] the present author introduced the concept of a *desynchronous domain* $\overline{\mathrm{SD}}$ – a set of samples of a time series satisfying the condition

$$\begin{aligned} &\overline{\mathrm{SD}}_r :\ \{J_{k'} = 0,\ J_{k''} = 1 \vee k'' = 0,\ J_{k'''} = 1 \vee k''' = K+1\}, \\ &k' \in \overline{b^{\overline{\mathrm{SD}}}_r,\, b^{\overline{\mathrm{SD}}}_r + L^{\overline{\mathrm{SD}}}_r - 1}, \quad k'' = b^{\overline{\mathrm{SD}}}_r - 1, \quad k''' = b^{\overline{\mathrm{SD}}}_r + L^{\overline{\mathrm{SD}}}_r, \end{aligned} \tag{4.24}$$

where $b^{\overline{\mathrm{SD}}}_r$, $L^{\overline{\mathrm{SD}}}_r$, and r are the emergence time, the length, and the ordinal number of a desynchronous domain $\overline{\mathrm{SD}}$, respectively.

The meaning of the characteristics introduced in this section is demonstrated in Figure 3 (see [16] for additional information).

4.1. Generalized T-synchronization

It follows from the definition of the synchronization condition (3.14) that the analyzer proposed evaluates the complete synchronization level [10] and detects antisynchronization [11] with lag synchronization [12] precisely in the alphabetic representation $\mathrm{T}^{\alpha\varphi}_o$. However, according to the definition of the geometry of the symbols of the T-alphabet (2.4), complete synchronization at the level of the samples of $T^{\alpha\varphi}_k$ is a wider phenomenon compared with the complete synchronization at the level of $\mathbf{s}_k$ – the samples of the sequence itself. The T-synchronism of dynamical systems (with respect to the set of phase variables $\mathbf{s}$) is considered from the viewpoint of the shape (geometric structure) of the trajectories of the systems in the extended phase space. By the shape (geometric structure) of a trajectory of a dynamical system in the extended phase space is meant its certain invariant under uniform translations and dilations of the trajectory in the space of phase variables.

Thus, in a sense, the T-synchronization deals with the topological aspects of synchronization of dynamical systems [20, 21]. Hence, this opens a possibility for the application of the analyzer proposed to the study of generalized synchronization of chaos [9].

To this end, we introduce two additions that relax the requirements imposed in Section 3.1 on the complete T-synchronization. The first is the rejection of the equality of symbols in (3.14), and the second is the rejection of the maximization of δ^s in (3.18).

The rejection of the equality of symbols in (3.14) allows one to proceed to the following definition.

Dynamical systems are *synchronous* at time instant k in the sense of Generalized T-synchronization if the condition $J_k = 1$ is satisfied, where

$$J_k = \begin{cases} 1 & T_k^{\alpha\varphi} \in \mathrm{M}^{\mathrm{JT}}, \\ 0 & \text{otherwise}. \end{cases}, \quad \mathrm{M}^{\mathrm{JT}} \subseteq \mathrm{T}_o^{\alpha\varphi}|N. \tag{4.25}$$

The structure of the set M^{JT} is not trivial. First, the cardinality of the set is

$$\left|\mathrm{M}^{\mathrm{JT}}\right| = \min\Bigl\{\left|\mathrm{V}^{\Gamma}|_1\right|, \, \ldots, \, \left|\mathrm{V}^{\Gamma}|_n\right|, \, \ldots, \, \left|\mathrm{V}^{\Gamma}|_N\right|\Bigr\}. \tag{4.26}$$

Second, the condition

$$\left|\mathrm{M}^{\mathrm{JT}}|_{i_n}\right| \leqslant 1, \quad \forall\, \mathtt{T}\, i_n \in \mathrm{V}^{\Gamma}|_n, \quad n \in \overline{1,\, N}, \tag{4.27}$$

is always satisfied.

As a rule, conditions (4.26) and (4.27) correspond to different variants of constructing the set M^{JT}. Let $\mathcal{M}^{\mathrm{JT}} \ni \mathrm{M}^{\mathrm{JT}}{}_j$ be the set of all admissible variants of constructing the set M^{JT}, where j is the number of a variant.

The number of variants of constructing the set M^{JT} is

$$\hat{N}^{\mathrm{JT}} = \prod_{i=0}^{P_N-1} \prod_{n=1}^{N-1} \Bigl(\left|\mathrm{V}^{\Gamma}|_n\right| - i\Bigr), \quad P_N = \left|\mathrm{V}^{\Gamma}|_N\right|. \tag{4.28}$$

Note that the indices n in (4.28) are rearranged as follows:

$$\left|\mathrm{V}^{\Gamma}|_1\right| \geqslant \ldots \geqslant \left|\mathrm{V}^{\Gamma}|_n\right| \geqslant \ldots \geqslant \left|\mathrm{V}^{\Gamma}|_N\right|. \tag{4.29}$$

If the condition

$$\left|\mathrm{V}^{\Gamma}|_1\right| = \ldots = \left|\mathrm{V}^{\Gamma}|_n\right| = \ldots = \left|\mathrm{V}^{\Gamma}|_N\right|,$$

is valid, then

$$\hat{N}^{\mathrm{JT}} = \prod_{i=0}^{P_N-1} \Bigl(P_N - i\Bigr)^{N-1} = \bigl(P_N\bigr)^N \frac{\Gamma^N\bigl(P_N\bigr)}{\Gamma\bigl(1 + P_N\bigr)},$$

where Γ is the gamma function.

It should also be noted that, in the general case, $\hat{N}^{\mathrm{JT}}$ represents an upper estimate, because the matrix of Δ^{T} may contain zero elements; i.e., not all possible combinations of elementary symbols are realized.

Thus, complete T-synchronization is a special case of generalized T-synchronization. Figure 4 illustrates the structure of the set M^{JT} for $N = 2$ in two cases of complete T-synchronization, the direct synchronization and antisynchronization.

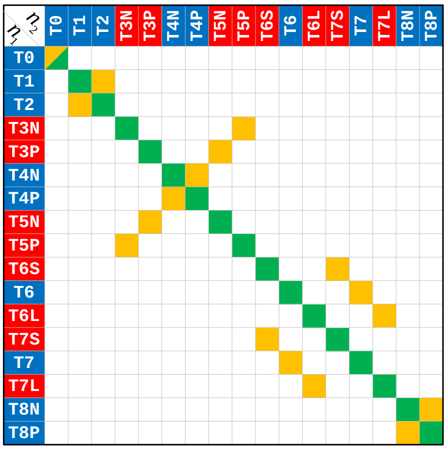

Figure 4. The structure of the set M^{JT} for $N = 2$ in two cases of complete T-synchronization: direct synchronization (green) and antisynchronization (orange).

According to (4.28), in the case of generalized T-synchronization, the problem arises of choosing a set M^{JT} from among the family $\mathcal{M}^{\mathrm{JT}}$ that is optimal with respect to some criterion. In the general case, this is a combinatorial optimization problem. Let us introduce a generalized algorithm

$$\mathrm{M}^{\mathrm{JT}} = \left\{ \mathrm{M}_i^{\mathrm{JT}} : \; F^{\mathrm{JT}} \to \max, \, \forall \, \mathrm{M}_i^{\mathrm{JT}} \in \mathcal{M}^{\mathrm{JT}} \right\}, \tag{4.30}$$

where F^{JT} is a objective function.

For the algorithm (4.30), we can consider the following basic objective functions. Maximization of the integral coefficient of synchronism

$$F_0^{\mathrm{JT}} = \frac{1}{K} \sum_{i=1}^{K} i \, H^{\mathrm{SD}} \left[i \right] \equiv \delta^s. \tag{4.31}$$

Maximization of the length of a synchronous domain

$$F_1^{\mathrm{JT}} = \max \left\{ L^{\mathrm{SD}} \left| H^{\mathrm{SD}} \left[L^{\mathrm{SD}} \right] \geq 1, \; L^{\mathrm{SD}} \in \overline{1, \; K} \right. \right\}. \tag{4.32}$$

Note that, if necessary, one can expand the set of objective functions. Moreover, one can impose additional constraints on condition (4.30).

A naive implementation of the algorithm (4.30) leads to difficulties of computational character. They are associated with combinatorial explosion. Let us illustrate this situation by an example of the full T-alphabet, $|\mathrm{T}_o^{\alpha\varphi}| = 17$:

$$\begin{aligned} N &= 2, \quad \hat{N}^{\mathrm{JT}} = 355\,687\,428\,096\,000, \\ N &= 3, \quad \hat{N}^{\mathrm{JT}} = 126\,513\,546\,505\,547\,170\,185\,216\,000\,000, \\ N &\gg 2, \quad \text{Curse of dimensionality!} \end{aligned}$$

Currently, the author has developed a suboptimal version of algorithm (4.30). The main idea of the approach is as follows: the algorithm operates only with those symbols from the set V^{Γ} for which the elements of the matrix Δ^{T} are close to the maximum value. Thus, this algorithm is free of the need of complete enumeration of the set $\mathcal{M}^{\mathrm{JT}}$; however, it does not guarantee the global optimum of the function F^{JT}. Note that this topic is the subject of our current research.

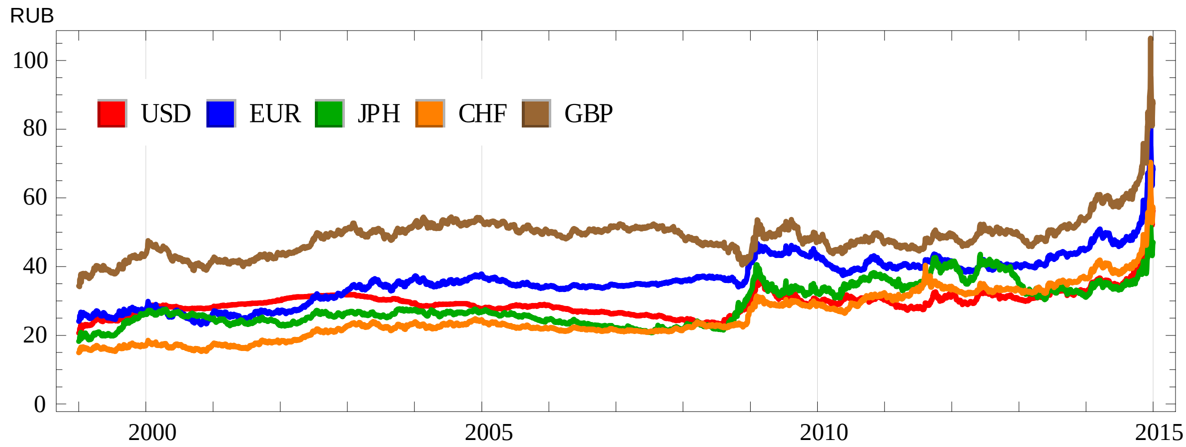

Figure 5. Currency exchange rates.

5. Sample

Let us demonstrate the capabilities of the tools developed by an example of the analysis of financial time series. The object of analysis is the time series of exchange rates of some world currencies (US dollar [USD], Euro [EUR], Japanese Yen [JPH], Swiss Franc [CHF], and British Pound [GBP] against Russian ruble). The analyzed period is from 01.01.1999 to 31.12.2014.

Note that the analysis of the Generalized T-synchronization has applied value in the context of research in macroeconomics and stochastic financial mathematics. The original data are taken from the official web-site of the Central Bank of Russia (Bank of Russia, exchange rates, www.cbr.ru/eng/). The length of the time series is $K = 3\,985$ samples. The initial time series are shown in Figure 5. Note that these time series have also been studied from the viewpoint of complete T-synchronization [26] (the analyzed period is from 01.01.1999 to 31.03.2013) and TQ-complexity [27].

We carried out a detailed analysis of generalized T-synchronization for the USD/EUR pair. For comparison, we also evaluated the characteristics of complete synchronization. The set M^{JT} was constructed for the objective function F_0^{JT}. As a result, we obtained the following values of the integral coefficient of synchronism:

$$\delta^s|C = 0.174492, \quad \delta^s|A = 0.219433, \quad \delta^s|G = 0.222948, \tag{5.33}$$

where $\delta^s|C$, $\delta^s|A$, and $\delta^s|G$ are complete, anti-, and generalized regimes of synchronization, respectively.

The results (5.33) show that the regime of generalized T-synchronization is characterized by the maximum value of the integral coefficient of synchronism. At the same time, from the structural point of view, this mode is characterized by the combination of direct- and anti-synchronism regimes (see Figure 6).

Taking account of the factor of generalized synchronization for the pair USD/EUR modifies the time structure of its T-synchronism (see Figure 7).

Figure 7 shows that taking account of the factor of generalized synchronization significantly reduces the maximum length of desynchronous domains. In this case, the mean length of synchronous domains also decreases.

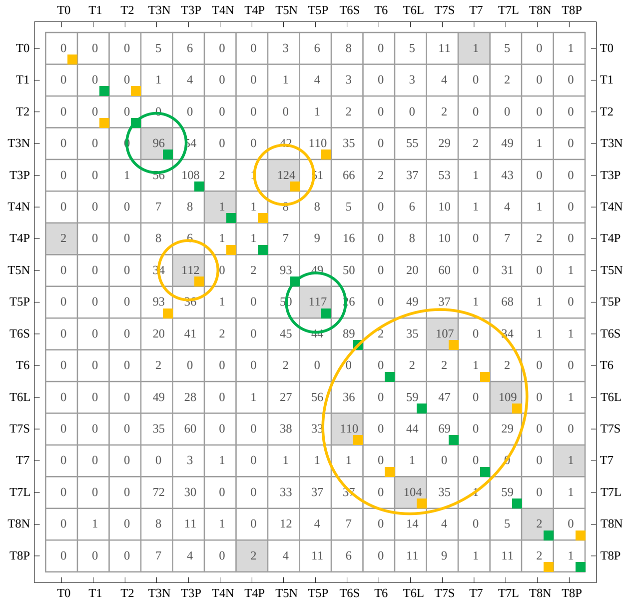

Figure 6. Matrix Δ^T; grey cells are T-symbols included in the set M^{JT}; green and orange squares demonstrate the structure of the set M^{JT} in two cases of complete T-synchronization (see Figure 4); green and orange ellipses are key T-symbols in the set M^{JT}.

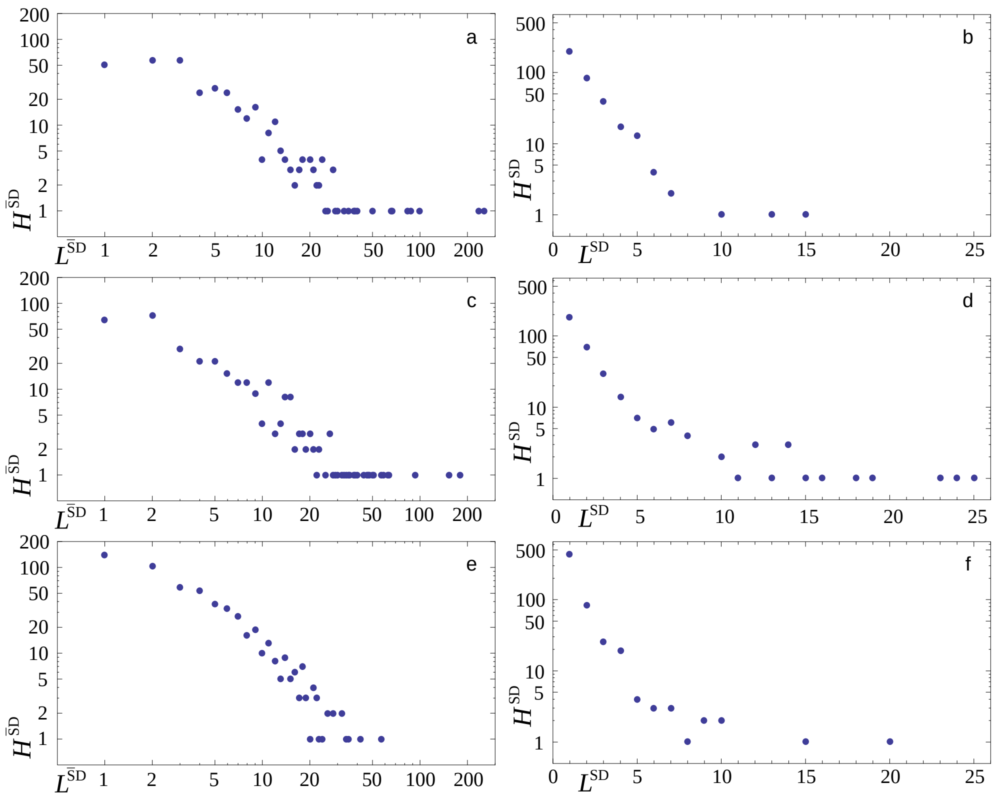

Figure 7. Spectral density of domains $\overline{\text{SD}}$ and SD; (a, b), (c, d), and (e, f) are complete, anti-, and generalized regimes of synchronization, respectively.

6. Conclusion

We have proposed a new method to diagnose generalized synchronization in nonlinear multidimensional chaotic systems. The method allows one to explore and quantitatively evaluate the time structure of synchronization of chaotic oscillations in the so-called T-synchronization regime [16, 17]. The approach is based on the formalism of symbolic CTQ-analysis, which was proposed by the author in [18, 19].

The method considered, which is based on the analysis of generalized T-synchronization, can be successfully applied to the study of multidimensional systems consisting of two or a greater number of coupled nonidentical oscillators, including multidimensional lattices of oscillators with arbitrary topology. The approach described can be applied to the analysis of experimental data, because it does not require any a priori knowledge of the system under study.

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