Estimation of the TQ-complexity of chaotic sequences

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Why is Complexity?

Complexity:

It is one of the fundamental scientific concepts.

It is important information-structural characteristic of any object.

A narrower notion of

complexity of dynamic process

is no exception.

Additional information is presented in the papers:

Kravtsov Yu. A. Randomness, determinateness, and predictability.
Soviet Physics Uspekhi (1989), **32**(5):434

Complexity, Science and Society,
Eds. Bogg J., Geyer R., Radcliffe Publishing, 2007.

Measurements of Complexity,
Eds. Petiti L., Vulpiari A., Lect. Notes in Phys., 1988, 314.

Computation of complexity – it is an open question...

However, along with this, definition and calculation the complexity of dynamic processes remains of methodologically open questions:

- 1877 Year – Ludwig Boltzmann introduced the notion of "entropy"
- R. Hartley and C. Shannon – gave entropy of sense information
- A.N. Kolmogorov and Y.G. Sinai – entropy generalized to the dynamical systems
- Nonlinear Dynamics – Lyapunov exponents, Kolmogorov entropy, S-parameter Klymontovich, and etc.
- A.N. Kolmogorov – an algorithmic approach to the concept of "complexity"
- Radio physics – a time-frequency criterion of complexity
- V.I. Arnold – calculation complexity of latticed sequences of the form $\mathbb{Z}_2 \times \mathbb{Z}$
- ...

Computation of complexity – it is an open question...

Many approaches have different critical restrictions:

- some do not allow you to measure complexity of a particular trajectory;
- some are very laborious to compute and interpret the results;
- some have energetical sense and do not have informational sense;
- some do not carry over on $\mathbb{R}^N \times \mathbb{Z}$-the continual process.
- ...

Who is to blame? What to do?

Problem statement

We introduce a discrete dynamical system:

$$\mathbf{s}_{k+1} = \mathbf{f}\left(\mathbf{s}_k,\, \mathbf{p}\right),$$

$$\mathbf{s} \in \mathrm{S} \subset \mathbb{R}^N, \quad k \in \mathrm{K} \subseteq \mathbb{Z}, \quad \mathbf{p} \in \mathrm{P} \subset \mathbb{R}^M, \quad n = \overline{1,\, N}, m = \overline{1,\, M}.$$

In addition, consider a discrete semi-sequence:

$$\{\mathbf{s}_k\}_{k=1}^{K}, \quad k \in \mathrm{K} = \overline{1,\, K},$$

it is observable trajectory of dynamical system.

Denote a space

$$\mathrm{S} \times \mathrm{K},$$

it is extended space of state of dynamical system.

How can we measure the complexity of the sequence $\{\mathbf{s}_k\}_{k=1}^{K}$ in the space $\mathrm{S} \times \mathrm{K}$?

Key idea of approach

Proposition

The more complex is a dynamic process, the more complex is the shape of its trajectory in the space $\mathrm{S} \times \mathrm{K}$.

This definition is ideologically close to perimetric complexity.

Perimetric complexity is a measure of the complexity of binary pictures. The concept of perimetric complexity was first introduced by:

F. Attneave and M.D. Arnoult, *The Quantitative Study of Shape and Pattern Perception* // Psychological Bulletin, 53 (6), 1956, pp. 452-471. http://psycnet.apa.org/journals/bul/53/6/452.

The symbolic CTQ-analysis is base to detection and to analysis shape of the trajectory in space $\mathrm{S} \times \mathrm{K}$ for sequence $\{\mathbf{s}_k\}_{k=1}^{K}$.

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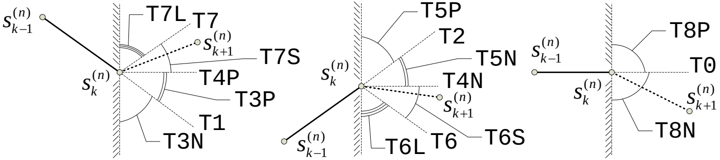
Definition of T-alphabet

We introduce the primary mapping:

$$\left\{\mathbf{s}_{k-1}^{(n)}, \mathbf{s}_{k}^{(n)}, \mathbf{s}_{k+1}^{(n)}\right\} \Rightarrow T_k^{\alpha\varphi}|_n, \quad T_k^{\alpha\varphi} = \left[T_k^{\alpha\varphi}|_1 \ldots T_k^{\alpha\varphi}|_N\right], \quad \{T_k^{\alpha\varphi}\}_{k=1}^{K},$$

where $T^{\alpha\varphi}|_n$ is symbol of T-alphabet:

$$\mathrm{T}_o^{\alpha\varphi} = \{\mathtt{T0}, \mathtt{T1}, \mathtt{T2}, \mathtt{T3N}, \mathtt{T3P}, \mathtt{T4N}, \mathtt{T4P}, \mathtt{T5N}, \mathtt{T5P}, \mathtt{T6S}, \mathtt{T6}, \mathtt{T6L}, \mathtt{T7S}, \mathtt{T7}, \mathtt{T7L}, \mathtt{T8N}, \mathtt{T8P}\}.$$



Thus occurs coding shaped of trajectory.

Definition of Q-alphabet

Additionally we define the $Q^{\alpha\varphi}|_n$ it is symbol of the Q-alphabet $\mathrm{Q}_o^{\alpha\varphi}$:

$$Q_k^{\alpha\varphi}|_n \equiv T_k^{\alpha\varphi}|_n \to T_{k+1}^{\alpha\varphi}|_n, \quad Q_k^{\alpha\varphi} = \left[Q_k^{\alpha\varphi}|_1 \dots Q_k^{\alpha\varphi}|_N\right], \quad \{Q_k^{\alpha\varphi}\}_{k=1}^{K}.$$

k \ $k+1$	T0	T1	T2	T3N	T3P	T4N	T4P	T5N	T5P	T6S	T6	T6L	T7S	T7	T7L	T8N	T8P
T0	■															■	■
T1		■		■	■		■						■	■	■		
T2			■			■		■	■	■	■	■					
T3N		■		■	■		■						■	■	■		
T3P		■		■	■		■						■	■	■		
T4N	■															■	■
T4P	■															■	■
T5N			■			■		■	■	■	■	■					
T5P			■			■		■	■	■	■	■					
T6S		■		■	■		■						■	■	■		
T6		■		■	■		■						■	■	■		
T6L		■		■	■		■						■	■	■		
T7S			■			■		■	■	■	■	■					
T7			■			■		■	■	■	■	■					
T7L			■			■		■	■	■	■	■					
T8N		■		■	■		■						■	■	■		
T8P			■			■		■	■	■	■	■					

Table of the transitions $T_k^{\alpha\varphi}|_n \to T_{k+1}^{\alpha\varphi}|_n$.
Admissible transitions are shown in green.

Symbolic TQ-image

Definition

The directed graph

$$\Gamma^{TQ}|_n = \left\langle \mathrm{V}^{\Gamma}|_n, \mathrm{E}^{\Gamma}|_n \right\rangle, \quad \mathrm{V}^{\Gamma}|_n \subseteq \mathrm{T}_o^{\alpha\varphi}, \quad \mathrm{E}^{\Gamma}|_n \subseteq \mathrm{Q}_o^{\alpha\varphi},$$

is a particular symbolic TQ-image of the dynamical system (discrete map) with respect to its n-th phase variable.

In the definition set $\mathrm{V}^{\Gamma}|_n$ is the vertex set and $\mathrm{E}^{\Gamma}|_n$ is the edge set of $\Gamma^{TQ}|_n$.

The graph $\Gamma^{TQ}|_n$ can be weighted (on its vertices and edges) by the occurrence frequency of characters $*$ in the sequence $\{\mathbf{s}_k^{(n)}\}_{k=1}^{K}$:

$$\Delta^*|_n = \frac{|\mathrm{M}^*|_n|}{\left|\bigcup_* \mathrm{M}^*|_n\right|}, \quad 0 \leqslant \Delta^*|_n \leqslant 1,$$

where $|\circ|$ is the cardinality of the set and $*$ is a symbol of which the multiset $\mathrm{M}^*|_n$ consists (T- or Q-symbols).

Extensions of symbolic CTQ-analysis

Symbolic CTQ-analysis has a number of extensions:

- CTQ-symmetry;
- TQ-bifurcations;
- TQ-complexity;
- T-synchronization;
- Q-prediction;
- Q-control.

Main articles (in English)

- A.V.M., *Structure of Synchronized Chaos Studied by Symbolic Analysis in Velocity–Curvature Space*. Technical Physics Letters, **38**:2 (2012), 155—159; arXiv: 1203.4214.
- A.V.M., *Multidimensional Dynamic Processes Studied by Symbolic Analysis in Velocity–Curvature Space*. Computational Mathematics and Mathematical Physics, **52**:7 (2012), 1017—1028.
- A.V.M., *Measure of Synchronism of Multidimensional Chaotic Sequences Based on Their Symbolic Representation in a T-Alphabet*. Technical Physics Letters, **38**:9 (2012), 804–808; arXiv: 1212.2724.
- A.V.M., *Analysis of the Time Structure of Synchronization in Multidimensional Chaotic Systems*. J. Exp. Theor. Phys., **120**:5 (2015), 912–921; arXiv: 1505.04314.

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Main conception

The TQ-complexity approach is based on two conceptions:

- First. Unit complexity of symbols T- and Q-alphabets.
- Second. Reduction operation on T-symbols sequence.

Unit complexity of the symbols

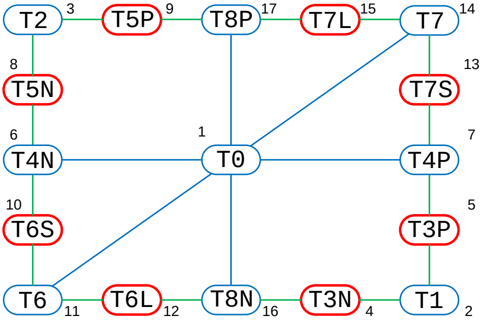
The unit complexity of the symbol $T^{\alpha\varphi}|_n$:

$T^{\alpha\varphi}\|_n$	T0	T1, T2	T4*, T8*	T3*, T5*	T6, T7	T6∘, T7∘
$C^T\|_n$	1	2	4	5	4	6

The unit complexity of the symbol $Q^{\alpha\varphi}|_n$:

$$C^Q|_n = \mathrm{d_T}\left(T_k^{\alpha\varphi}|_n,\, T_{k+1}^{\alpha\varphi}|_n\right) + 1,$$

where $\mathrm{d_T}(\cdot,\cdot)$ is the shortest path between two vertices in the graph $\mathrm{D_T^{1p}}$:



The graph $\mathrm{D_T^{1p}}$ corresponds to transitions between the symbols $T^{\alpha\varphi}|_n$ for the k-th sample of the subsequence $\left\{s_{k-1}^{(n)}, s_k^{(n)}, s_{k+1}^{(n)}\right\}$ under its various continuous one-point deformations.

Reduction operation

Assertion

When calculating the TQ-complexity of the sequence $\{T_k^{\alpha\varphi}|_n\}_{k=1}^{K}$, one should first reduce it; i.e., one should remove repeated subsequences, because they do not carry new information.

Indeed, consider two test sequences (letter T is omitted):

7S 5P 6L 7S 5P 6L 7S 5P 6L 7S 5P 6L 7S 5P 6L 7S . . . ,

7S 6L 7S 5P 5P 5P 6L 7S 5P 6L 7S 5P 6L 7S 6L 7S . . . .

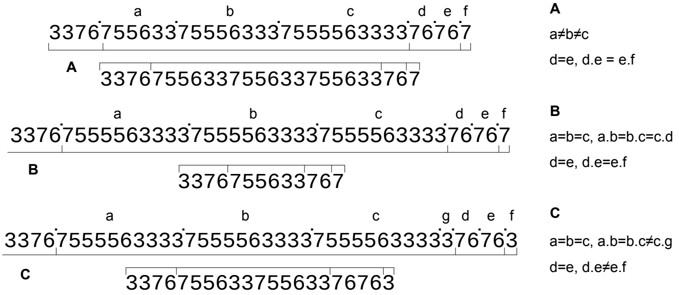
It should be noted the following statements:

- Intuitively and logically, the first sequence is simpler. It shows clear periodicity, which can easily be continued.
- The second sequence is objectively more complicated. It is not so easy to continue (the principle of its generation is unclear).
- At the same time, both sequences have the same set of symbols and differ only in their sequence order.

Reduction operation

Main principles of reduction operation:

- The removal of duplicate fragments is performed starting from longer to shorter ones.
- Furthermore, the removal of identical fragments is performed so that, locally (within the deleted blocks), the set of $T^{\alpha\varphi}|_n$ and $Q^{\alpha\varphi}|_n$ symbols is preserved. This guarantees the invariance of the graph $\Gamma^{TQ}|_n$.
- After the application of this rule, we obtain a reduced sequence $\{T_k^{\alpha\varphi}|_n\}_{k=1}^{K^*}$.



This operation allows one to distinguish between periodic and chaotic segments.

Degenerate and Weighted measures of complexity

We introduce two measures of TQ-complexity:

The Degenerate measure $\mathbf{C}_S^{\mathrm{d}} = \left[C_{ST}^{\mathrm{d}},\, C_{SQ}^{\mathrm{d}}\right]^{\mathrm{T}}$.

Formally, the unit complexities of the symbols $T^{\alpha\varphi}|_n$ and $Q^{\alpha\varphi}|_n$ are assumed to be equal to unity:

$$C_{ST}^{\mathrm{d}}|_n = K^*, \quad C_{SQ}^{\mathrm{d}}|_n = K^* - 1,$$

where K^* is length of reduced sequence.

The Weighted measure $\mathbf{C}_S^{\mathrm{w}} = \left[C_{ST}^{\mathrm{w}},\, C_{SQ}^{\mathrm{w}}\right]^{\mathrm{T}}$.

That is, the weighted measure also includes the unit complexities of the symbols $T^{\alpha\varphi}|_n$ and $Q^{\alpha\varphi}|_n$:

$$C_{ST}^{\mathrm{w}}|_n = \sum_{k=1}^{K^*} C^T|_n \left[T_k^{\alpha\varphi}|_n\right], \quad C_{SQ}^{\mathrm{w}}|_n = \sum_{k=1}^{K^*-1} C^Q|_n \left[T_k^{\alpha\varphi}|_n,\, T_{k+1}^{\alpha\varphi}|_n\right].$$

Also, We introduced the Spectrum of Reductions $H^{\mathrm{DS}}\left[L^{\mathrm{DS}},\, N^{\mathrm{DS}}\right]$, where H^{DS} is the number of acts of reduction, L^{DS} is the length of a subsequence to be reduced, and N^{DS} is the number of removed fragments.

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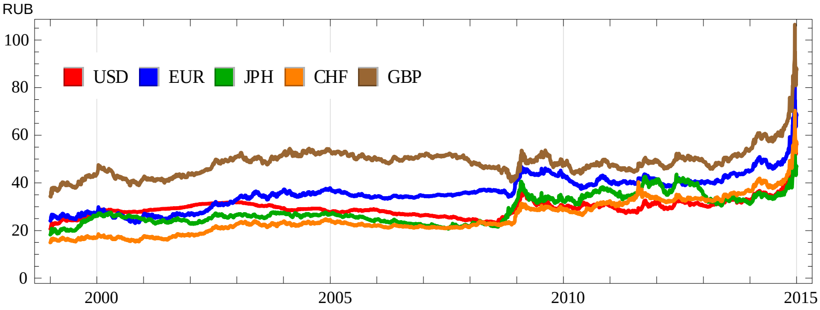
Exchange rates of some world currencies

The object of analysis is the time series of exchange rates of some world currencies (US dollar [USD], Euro [EUR], Japanese Yen [JPH], Swiss Franc [CHF], and British Pound [GBP] against Russian ruble).

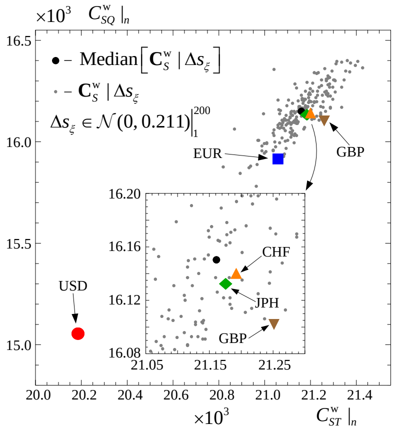


The analyzed period is from 01.01.1999 to 31.12.2014.

The original data are taken from the official web-site of the Central Bank of Russia (Bank of Russia, exchange rates, www.cbr.ru/eng/).

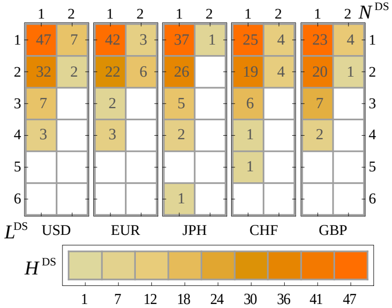
TQ-complexity of the time series

Weighted measure



Complexity of the pair USD/RUB is much lower than the complexity of the other sequences (including random seq).

Spectrum of Reductions



The JPH/RUB pair contains single removed fragment with a length of 6 T-symbols.

Information about the spectrum of reductions may also be useful for the analysis of the short-term predictability of currency exchange rates.

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Summary

- In this report, we have proposed a new approach to the quantitative evaluation of the complexity of multidimensional chaotic sequences. The algorithm is based on the method of symbolic CTQ-analysis.
- This approach is free from most of the disadvantages of existing methods for estimating the complexity of dynamic processes.
- This algorithm operates not only with the frequency of occurrence of symbols, but also takes into account the sequence order of the symbols.
- Note that, according to their design, the measures of TQ-complexity are directly related to such issues as periodic orbits, entropy of a dynamical system, etc. The central element of this relation is the spectrum of the reductions $H^{\mathrm{DS}}\left[L^{\mathrm{DS}}, N^{\mathrm{DS}}\right]$.

  A detailed study of this relation is the subject of our future research.

Thank you for your attention!